CSCI 33500 - Spring 2016 -
Answers to Homework #1, Covering chapters 1 and 2.

1 Chapter 1 - Mathematical Background

1.1 Prove by induction that $\sum_{i=1}^{N-2} F_i = F_N - 2$
where F_i is the i-th Fibonacci number, as defined in section 1.2 / page 6 of the book.

Answer: Base case: For $N = 3$, we have $\sum_{i=1}^{3-2} F_i = F_1 = 1$ and $F_3 - 2 = 3 - 2 = 1$
Induction: Assuming that the equality is true for a given N, we have

$$\sum_{i=1}^{(N+1)-2} F_i = F_{N-1} + \sum_{i=1}^{N-2} F_i$$
$$= F_{N-1} + F_N - 2 \text{ by the induction hypothesis}$$
$$= F_{N+1} - 2 \text{ by the definition of the Fibonacci series}$$

Q.E.D.

Thus the equality is true for all $N \geq 3$.

1.2 Prove by induction that $\sum_{i=1}^{N} i^3 = (\sum_{i=1}^{N} i)^2$

Answer: Base case: For $N = 1$, we have $\sum_{i=1}^{1} i^3 = 1^3 = 1$ and $(\sum_{i=1}^{N} i)^2 = 1^2 = 1$
Induction: Assuming that the equality is true for a given N, we have

$$\sum_{i=1}^{N+1} i^3 = (N+1)^3 + \sum_{i=1}^{N} i^3$$
$$= (N+1)^3 + (\sum_{i=1}^{N} i)^2 \text{ by the induction hypothesis}$$
$$= (N+1)^3 + \left(\frac{N(N+1)}{2}\right)^2$$

(we know the formula for the sum of the first N integers)

$$= (N+1)^3 + \frac{N^2(N+1)^2}{4}$$

$$= (N+1)^2 \left((N+1) + \frac{N^2}{4} \right) \text{ by factoring } (N+1)^2$$

$$= (N+1)^2 \left(\frac{4N+4+N^2}{4} \right)$$

$$= (N+1)^2 \left(\frac{(N+2)^2}{2^2} \right)$$

$$= \left(\frac{(N+1)(N+2)}{2} \right)^2$$

$$= \left(\sum_{i=1}^{N+1} i \right)^2 \text{ (sum of first N+1 integers)}$$

Q.E.D.

Thus the equality is true for all $N \geq 1$.

1.3 Prove that $2^{99} \equiv 1 \pmod 7$

Answer: $2^3 = 8 \equiv 1 \pmod 7$ thus $2^{99} = (2^3)^{33} \equiv 1^{33} \pmod 7$ thus $2^{99} \equiv 1 \pmod 7$
Rule used: $A \equiv B \bmod (M)$ implies $A^X \equiv B^X \bmod (M)$.

2 Chapter 2 - Algorithm Analysis

2.1 Order the following functions by growth rate: $N, \sqrt{N}, N^{1.5}, N^2, N \log N, N \log \log N$
$N(\log N)^2, N \log N^2, 2/N, 2^N, 2^{N/2}, 99$ (constant), $N^2 \log N, N^3, N^N, N!$.
If two functions grow at the same rate, indicate so.

Answer: from slowest to fastest growing $2/N, 99, \sqrt{N}, N, N \log \log N, N \log N, N \log N^2$,
$N(\log N)^2, N^{1.5}, N^2, N^2 \log N, N^3, 2^{N/2}, 2^N, N!, N^N$
$N \log N$ and $N \log N^2$ growth at the same rate.

2.2 Find two function $f(N)$ and $g(N)$ such that neither $f(N) = O(g(N))$ nor $g(N) = O(f(N))$. Explain your answer.

Answer: There are many possible answers. Here is an example:
$f(N) = N$ and $g(N) = N^2 * \cos(N)^2$
$\cos(N)^2$ varies between 0 and 1 periodically, thus $N^2 * \cos(N)^2$ varies from 0 to N^2 over each period of cos.
Thus $\lim_{N \to +\infty} \frac{f(N)}{g(N)}$ does not exist

2.3 Give a Big-O analysis of the running time of the following code:

```
sum = 0;
for(i=0; i<N; ++i)
    for(j=0; j<i*i; ++j)
       for(k=0; k<j; ++k)
           ++sum;
```

Answer: j can be as large as i^2, which could be as large as N^2. k can be as large as j, which is N^2. The running time is thus proportional to $N * N^2 * N^2$, which is $O(N^5)$. Alternatively:

$$\begin{aligned}
T(N) &= \sum_{i=0}^{N-1}\left(\sum_{j=0}^{i^2-1}\left(\sum_{k=0}^{j-1} 1\right)\right) \\
&= \sum_{i=0}^{N-1}\left(\sum_{j=0}^{i^2-1} j\right) \\
&= \sum_{i=0}^{N-1} \frac{i^2 * (i^2-1)}{2} \\
&= \frac{1}{2} * \left(\sum_{i=0}^{N-1} i^4 - \sum_{i=0}^{N-1} i^2\right) \\
&= O(N^5 - N^3) = O(N^5)
\end{aligned}$$

3 Extra Credit

Give a Big-O analysis of the running time of the following code:

```
sum = 0;
for(i=0; i<N; ++i)
    for(j=0; j<i*i; ++j)
        if (j%i == 0)
            for(k=0; k<j; ++k)
                ++sum;
```

Compare this to the running time of the algorithm in question 2.3

Answer: The if statement is executed at most N^3 times, by previous arguments, but it is true only $O(N^2)$ times (because it is true exactly i times for each i). Thus the innermost loop is only executed $O(N^2)$ times. Each time through, it takes $O(j) = O(i^2) = O(N^2)$

time, for a total of $O(N^4)$. This is an example where multiplying loop sizes can occasionally give an overestimate.